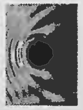


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In the last two decades, fractional (or non integer) differentiation has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, control theory and signal, and image processing. For example, in the last three fields, some important considerations such as modelling, curve fitting, filtering, pattern recognition, edge detection, identification, stability, controllability, observability and robustness are now linked to long-range dependence phenomena. Similar progress has been made in other fields listed here.

A fractional order system is a system described by an integro-differential equation involving fractional order derivatives of its input(s) and/or output(s). From a physical point of view, linear fractional derivatives and integrals order systems are not quite conventional linear systems, and not quite conventional distributed parameter systems. They are in fact halfway between these two classes of systems, and are particularly suited for diffusion phenomena modelling. They also have been a modelling tool well suited to a wide class of phenomena with non-standard dynamic behaviour.

The scope of *Fractional order systems* is thus to present some recent results in the study of fractional systems and the application of fractional differentiation. The 27 articles are grouped in five major areas:

- analytical and numerical solutions and approximation of fractional order systems;
- fractional order systems properties analysis;
- applications in system modelling;
- applications in system identification;
- applications in control theory and robotics.

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Fractional order systems

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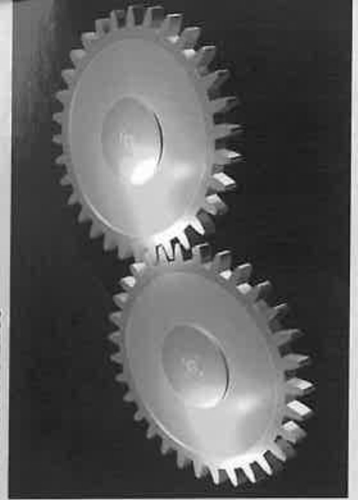

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Fractional order systems

Applications in modelling, identification and control

Guest Editors

Jocelyn Sabatier
Pierre Melchior
José A. Tenreiro Machado
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Hermes
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Pseudo phase plane, delay and fractional dynamics

Miguel Francisco Martins Lima*
José Antonio Tenreiro Machado**
Manuel Marques Crisóstomo***

** Dept. of Electrical Engineering Superior School of Technology*
Polytechnic Institute of Viseu, 3504-510 Viseu, Portugal
lima@mail.estv.ipv.pt

*** Dept. of Electrical Engineering, Institute of Engineering*
Polytechnic Institute of Porto, 4200-072 Porto, Portugal
jtm@isep.ipp.pt

**** Institute of Systems and Robotics*
Dept. of Electrical and Computer Engineering, University of Coimbra
Polo II, 3030-290 Coimbra, Portugal
mcris@isr.uc.pt

ABSTRACT. This paper analyses the relationship between the signal delay and the fractional dynamics. It is shown that some experimental signals spectra are approximated by trendlines. Based on the slope of these trendlines the fractional or integer order behavior is determined. For the pseudo phase plane reconstruction of each signal, the time delays are calculated through the fractal dimension, as an alternative to the mutual information that is often used. The slopes of the trendlines spectra reveal a relationship with the fractal dimension of the pseudo phase plane and the corresponding time lag.

RÉSUMÉ. Cet article analyse le rapport entre le retard de signal et la dynamique fractionnaire. Le spectre de quelques signaux robotiques est approximé par des lignes de tendance. De l'inclinaison de ces lignes, on considère son comportement comme d'ordre fractionnaire ou d'ordre entier. Pour la reconstruction du pseudo-plan de phase de chaque signal, les retards de temps sont calculés par la dimension fractale au lieu du calcul plus usuel par l'information mutuelle. Les inclinaisons des lignes de tendance du spectre montrent un rapport avec la dimension fractale du pseudo-plan de phase et du retard correspondant.

KEYWORDS: pseudo-phase plane, fractional calculus, mutual information, fractal dimension, robotics.

MOTS-CLÉS : pseudo-plan de phase, calcul fractionnaire, information mutuelle, dimension fractale, robotique.

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1. Introduction

The study of fractional order systems received considerable attention recently (Machado, 2003), due to the facts that many physical systems are well characterized by fractional models (Podlubny, 2002). With the success in the synthesis of real noninteger differentiators, the emergence of new electrical elements (Bohannan, 2002), and the design of fractional controllers (Sabatier *et al.*, 1998; Melchior *et al.*, 2000; Machado, 1997; Barbosa *et al.*, 2004), fractional calculus (FC) have been applied in a variety of dynamical processes (Oustaloup *et al.*, 1997; Vinagre *et al.*, 2002). The importance of fractional order mathematical models is that it can be used to make a more accurate description and to give a deeper insight into the physical processes underlying long range memory behaviors. On previous works (Lima *et al.*, 2006; Lima *et al.*, 2007) it was demonstrated that some robotic signals have a fractional behavior and constitute a good test-bed for the study of these phenomena.

A delay differential equation (DDE) (Driver, 1977; Faybishenko, 2004; Deng *et al.*, 2007) is a description where the evolution of a system at a certain time t, depends on the state of the system at an earlier time t–T. On the other hand, the FC incorporates a memory-time property because it captures the dynamic phenomena involved during all the time-history of a system (Le Méhauté *et al.*, 1991; Nigmatullin, 2006). Consequently, it seems to exist some kind of relationship between the FC and the integer models with delays, since both are based in memory aspects. This work is a first step towards the analysis of the hypothetical relationship between the DDE and the FC.

The pseudo phase space (PPS) is used to analyze signals with nonlinear behavior. For the two-dimensional case it is called pseudo phase plane (PPP) (Feeny *et al.*, 2004; Abarbanel *et al.*, 1993; Trendafilova *et al.*, 2001). To reconstruct the PPS it is necessary to find the adequate time lag between the signal and one delayed image of the original signal. To determine the proper time delay often the mutual information concept is used. Nevertheless, in some cases the mutual information reveals a behavior where it becomes difficult to find the adequate time delay. Alternatively, a method based on the fractal dimension to determine the proper delay is proposed in this paper. Some recent research addressed the relationship between fractal dimensions and fractional models (Novikov *et al.*, 2000; Koga *et al.*, 2004) but, both theoretical and experimental evidences are still to be explored further.

Bearing these ideas in mind, this paper is organized as follows. Section 2 describes the robotic system used to captured the signals. Sections 3 and 4 present some fundamental concepts, and the experimental results, respectively. Finally, Section 5 draws the main conclusions and points out future work.

2. Experimental platform

In order to analyze signals that occur in a robotic manipulator an experimental platform was developed. Due to the multiplicity of sensors, the data can be redundant because the information flows through all system. Therefore, the aim of the research work is to study the sensor signals in a redundancy perspective, and their dependence on the system dynamics.

The experimental platform has two main parts: the hardware and the software components (Lima *et al.*, 2005). The hardware architecture is shown in Figure 1. Essentially it is made up of a mechanical manipulator, a PC and an interface electronic system. The interface box is inserted between the arm and the robot controller, in order to acquire the internal robot signals; nevertheless, the interface captures also external signals, such as those arising from accelerometers and force/torque sensors. The modules are made up of electronic cards specifically designed for this work. The function of the modules is to adapt the signals and to isolate galvanically the robot's electronic equipment from the rest of the hardware required by the experiments.

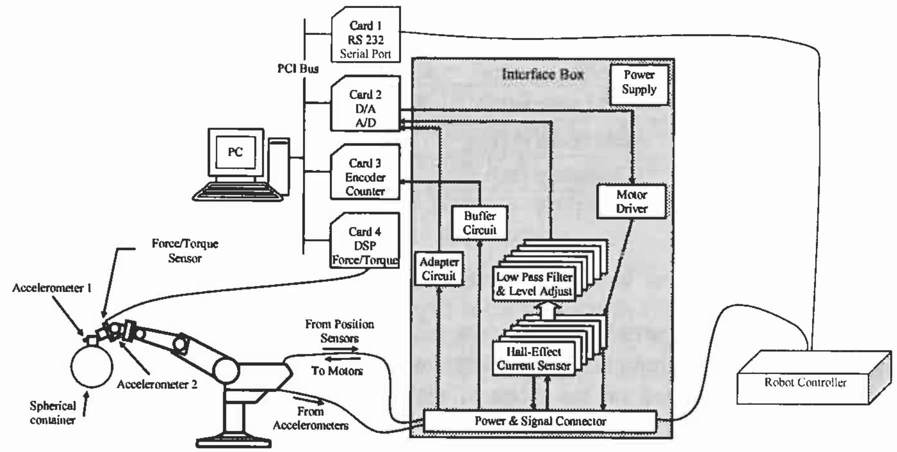


Figure 1. *Block diagram of the hardware architecture*

The software package runs in a Pentium 4, 3.0 GHz PC and, from the user's point of view, consists of two applications: (*i*) the acquisition application is a real time program responsible for acquiring and recording the robot signals; (*ii*) the analysis package runs off-line and handles the recorded data. This program allows several signal processing algorithms such as, Fourier transform (FT), Windowed FT, correlation, time synchronization, etc.

A spherical container carrying a liquid is adopted to test the load dynamics. To analyze the behavior of the variables in different situations, the container (Figure 2) can remain empty or can be filled with a liquid or a solid. The corresponding physical properties are shown in Table 1. The robot motion is programmed in a way that the container moves from an initial to a final position following a linear trajectory. The signals from different sensors, such as accelerometers, force and torque sensor, position encoders and current sensors are recorded with a sampling period of $t_s = 2 \times 10^{-3}$ s during a total time of $t_T = 20$ s.

Figure 2. *Spherical container with liquid*

Table 1. *Physical properties of the spherical container*

Characteristic	Spherical container
Mass (empty) [kg]	215×10^{-3}
Diameter [m]	203×10^{-3}

3. Main concepts

In this section is presented a review of fundamental concepts involved in the experiments. The technique used to determine the fractional behavior of several robotic signals is based on the slope of their spectra trendlines. Additionally, the pseudo phase space is obtained using the method of the time delays.

3.1. *Spectrum trendlines*

In order to examine the behavior of the signal FT a trendline is superimposed over the signal spectrum in, at least, one decade. The trendline is based on a power law approximation (Lima *et al.*, 2006):

$$|\mathrm{F}\{f(t)\}| \approx c\omega^m \quad [1]$$

where F is the Fourier operator, ω is the frequency, $c \in \Re^+$ is a constant that depends on the signal amplitude, and $m \in \Re$ is the slope. According to the value of m the signals can exhibit an integer or fractional order behavior.

We must mention that not all signals are possible to approximate through expression [1]. In fact, some signals present a scattered spectrum, being difficult to characterize using a simple analytical expression (Lima *et al.*, 2007). Therefore, model [1] is adopted for the signals following approximately that behavior, because it leads to a simple method of comparison.

3.2. *Pseudo phase plane and its reconstruction*

In the experimental study of the dynamical phenomena usually is not possible to sense all the states in a system. The PPS reconstruction mitigates this lack of information about the system. The goal of the PPS reconstruction is to view the signal in a higher dimensional space taking a sample measurement of its history. In order to achieve the phase space the proper time lag T_d for the delay measurements and the adequate dimension $d \in \mathrm{N}$ of the space must be determined. In the PPP the measurement $s(t)$ forms the *pseudo* vector $y(t)$ according to:

$$y(t) = [s(t), s(t+T_d), \ldots, s(t+(d-1)T_d)] \quad [2]$$

The vector $y(t)$ can be plotted in a d-dimensional space forming a curve in the PPS. There is a one-to-one relationship between the data in the PPS and the associated data in the true state space. If $d = 2$ we have a two-dimensional time delay space. Therefore, the plot of PPP will not change substantially, since the signal $\{s(t), s(t+T_d)\}$ is related with the model $\{s(t), \dot{s}(t)\}$. In resume, we expect the PPP to preserve the major properties of the state space representation, and thus to allow us to draw conclusions about the system dynamics.

The procedure of choosing a sufficiently large d is formally known as embedding, and any dimension that works is called an embedding dimension d_E. The number of measurements d_E should provide a phase space dimension, in which the geometrical structure of the plotted PPS is completely unfold, and where there are no hidden points in the resulting plot.

3.3. *Determining the time delays*

If we choose T_d too small, then the time series $s(t)$ and $s(t+T_d)$ will be so close to each other (in numerical value) that we cannot distinguish them. From a practical point of view they have not provided us with two independent coordinates. Similarly, if T_d is too large, then $s(t)$ and $s(t+T_d)$ are completely independent of each other (in a statistical sense) and the resulting time series present totally unrelated directions.

Among others (Feeny *et al.*, 2004), the method of delays is the most common method for reconstructing the phase space. Several techniques have been proposed to choose an appropriate time delay (Abarbanel *et al.*, 1993; Choi *et al.*, 1996). One line of thought is to choose T_d based on the correlation R of the time series with its delayed image. The correlation provides a measurement of the similarity between two time series that leads to good results when the series have a linear relationship. When R is minimal it indicates that the delay T_d will lead to the independency of the series $s(t)$ and $s(t+T_d)$. The difficulty of correlation to deal with nonlinear relations leads to the use of the mutual information. This concept from the information theory (Spataru, 1970) recognizes the non-linear properties of the series and measures their dependence. The mutual information for the two series of variables t and $t+T_d$ is given by:

$$I(t,t+T_d)=\log_2\frac{F_1\{s(t),s(t+T_d)\}}{F_2\{s(t)\}F_3\{s(t+T_d)\}} \qquad [3]$$

where $F_1\{s(t), s(t+T_d)\}$ is as a bidimensional probability density function and $F_2\{s(t)\}$, $F_3\{s(t+T_d)\}$ are the marginal probability distributions of the two series $s(t)$ and $s(t+T_d)$, respectively.

The average mutual information between the two time series is given by:

$$I_{av}(t,t+T_d)=\int_t\int_{t+T_d} F_1\{s(t),s(t+T_d)\}\log_2\frac{F_1\{s(t),s(t+T_d)\}}{F_2\{s(t)\}F_3\{s(t+T_d)\}}dtd(t+T_d) \qquad [4]$$

The index I_{av} allows us to obtain the time lag required to construct the pseudo phase space. To find the best value T_d of the delay, I_{av} is computed for a range of delays and the first minimum is chosen (widely referred in the literature thought not clearly). This procedure leads to the selection of the delay T_d corresponding to two time series that have a minimal mutual information, and, hence, to an optimal independence without excessively large delays. Usually I_{av} is referred (Feeny *et al.*, 2004; Abarbanel *et al.*, 1993; Trendafilova *et al.*, 2001) as the preferred alternative to select the proper time delay T_d. Nevertheless, practice reveals that in some cases is difficult to find the first minimum of I_{av} due to a noise or, even, because it does not exist. On other hand, the mutual information depends on the number of bins (classes) C adopted to calculate the probability density function $F_1\{s(t), s(t+T_d)\}$. Due to those issues an alternative is proposed to select the best delay T_d based on the fractal dimension of the PPP.

The fractal dimension is defined as:

$$\lim_{\varepsilon\to 0}\frac{\ln[N(\varepsilon)]}{\ln(1/\varepsilon)} \qquad [5]$$

where $N(\varepsilon)$ represents the minimal number of covering cells (*e. g.*, boxes) of size ε required to cover a set S. The slope on a plot of $\ln[N(\varepsilon)]$ versus $\ln(1/\varepsilon)$ provides an estimate of the fractal dimension.

The tests developed in this article prove that the fractal dimension of the PPP (dim_{PPP}) *versus* the time delay has a maximum corresponding to an adequate value for the chart construction. It should be mentioned that no theoretical proof is

provided. The relation between fractal dimension and time delay is only deduced from practical experiments. The same perspective motivates the links between fractal dimension and fractional order, and between fractional dynamics and long memory behavior.

4. Results

According to the platform described in Section 2 we adopt an experiment related with the internal movement of the liquid container. Several signals are captured in this experiment and the fractional behavior *versus* the PPP reconstruction is analyzed.

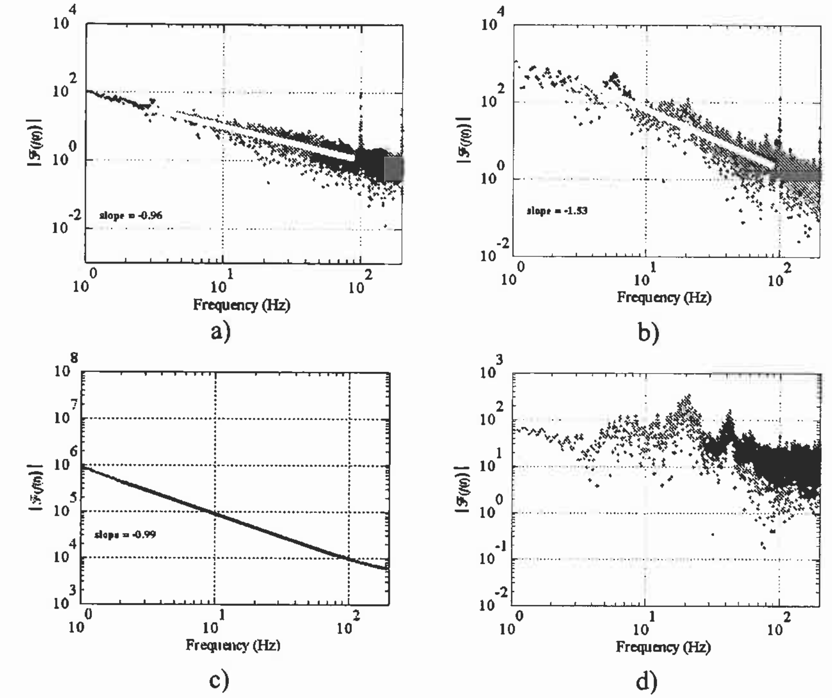
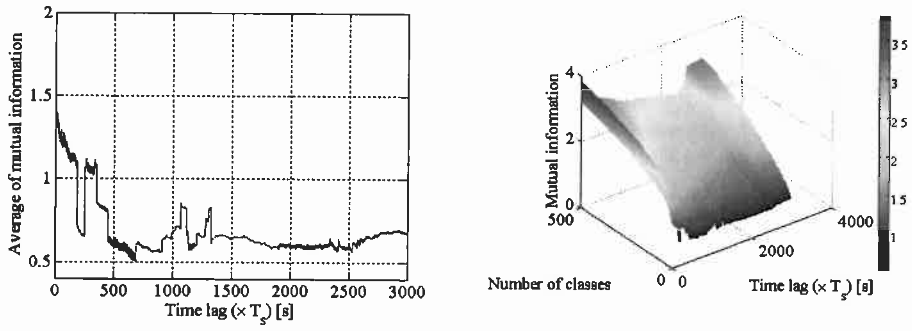


Figure 3. *The amplitude of the FFT for the experiment with a liquid container: a) electrical current of robot axis 5 (container filled with a liquid); b) electrical current of robot axis 3 (empty container); c) axis 1 position (empty container); d) acceleration at the clamped end of the container (container filled with a liquid)*

Figure 3a shows the amplitude of the FFT of the electrical current of robot axis 5 for the container filled with a liquid. A trendline is calculated and is superimposed over the signal in a frequency range larger than one decade ($3 < f < 90$ Hz). Its slope yields $m = -0.96$, revealing, clearly, the integer order behavior. Figure 3b shows the amplitude of the FFT of the electrical current of robot axis 3 for the empty container case. Again a trendline is calculated, and is superimposed over the signal in the same frequency range. Its slope yields $m = -1.53$, typical of a fractional order behavior. According to the manufacturer specifications the loop control of the robot has a cycle time of $t_c = 10$ ms. This fact is observed approximately at the fundamental ($f_c = 100$ Hz) and multiple harmonics in all spectra of motor currents.

Figure 3c shows the amplitude of the FFT axis 1 position signal (empty container). A trend line is calculated and is superimposed over the spectrum, with slope $m = -0.99$, that is a integer order behavior. Figure 3d shows the amplitude of the FFT of the acceleration of the container filled with a liquid. This spectrum is not so well defined in a large frequency range. All acceleration and the force/moments spectra present identical behavior. Therefore, it is difficult to define accurately the behavior of those signals in terms of integer or fractional system.

In resume, the electrical currents of the axis motors and the position axis seem to be well defined and constitute good candidates for being approximated through trendlines.

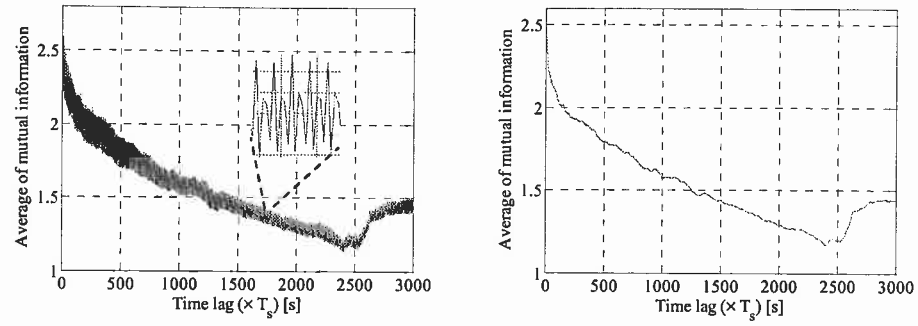


a) b)

Figure 4. *The index I_{av} of the electrical current of robot axis 2 for the experiment with the liquid container empty: a) I_{av} vs time lag for $C = 10$; b) I_{av} vs time lag and number of classes C*

Figure 4a depicts I_{av} of the electrical current of robot axis 2 for the experiment with the liquid container empty, when the number of classes is $C = 10$, revealing considerable discontinuities. As referred previously, the average mutual information I_{av} depends on the value of C adopted to calculate $F_1\{s(t),s(t+T_d)\}$. This fact can be seen in Figure 4b). The time lag T_d (one sample corresponds to 2×10^{-3} s) and the number of classes C vary in the ranges $0 < T_d < 3000$ and $10 < C < 500$, respectively. If C is too small I_{av} presents some discontinuities. Nevertheless, the larger the C the greater the processing time. For a given time lag T_d, I_{av} presents a monotonic curve when C varies. Bearing these ideas in mind, it was adopted $C = 100$ because it represents a useful compromise in terms of processing time.

Figure 5 depicts I_{av} of the electrical current of robot axis 2 for the experiment with the liquid container empty and $C = 100$, revealing an oscillatory behavior (see the zoom in Figure 5a). It was verified that I_{av} always presents a certain degree of noise/oscillation and that its amplitude, in general, is smaller when compared with the one of Figure 4a. Consequently, in order to use I_{av} an algorithm must be envisaged for smoothing the curve. Figure 5b depicts a smooth version of I_{av} when a minimum least squares algorithm is used, but it is not obvious the choice of the adequate minimum because several local minima occur. Due to these issues an alternative is proposed to select the best delay T_d based on the dim_{PPP}.



a) b)

Figure 5. *The index I_{av} vs time lag of the electrical current of robot axis 2 for the experiment with the liquid container empty for $C = 100$: a) original; b) smoothed version*

Figure 6a depicts I_{av} (smoothed version) of the electrical current of robot axis 3 for the experiment with the liquid container empty when $C = 100$. The first local minimum occurs for time lag $T_d = 150$ (0.3 s). In brackets is shown the delay in seconds corresponding to the 150 consecutive positions of the time series captured with a sampling frequency of $f_s = 500$ Hz. The second evident local minimum occurs for time lag $T_d = 525$ (1.05 s). As referred before, in some cases, it is not obvious the choice of the adequate minimum. Figure 6b depicts dim_{PPP} *versus* the time lag for the same robotic signal. The step adopted for the time lag was 25 (0.05 s) because it represents a useful compromise between the processing time and the resolution. The index dim_{PPP} presents a first local maximum for $T_d = 150$ (0.3 s), and a global maximum for $T_d = 525$ (1.05 s). Figures 6c, 6d depict the PPP for two different time lags of the electrical current of robot axis 3 for the experiment with the liquid container empty. Several experiments revealed that the adequate PPP is the one corresponding to Figure 6d because it is the most unfolded one. Figures 6e, 6f represent the corresponding 3-dimensional PPS where is visible that the unfolded parts of the PPP remains unfolded. Identical behaviors have the folded parts. Therefore, it seems reasonable to use the fractal dimension of the PPP to gather properties of the PPS.

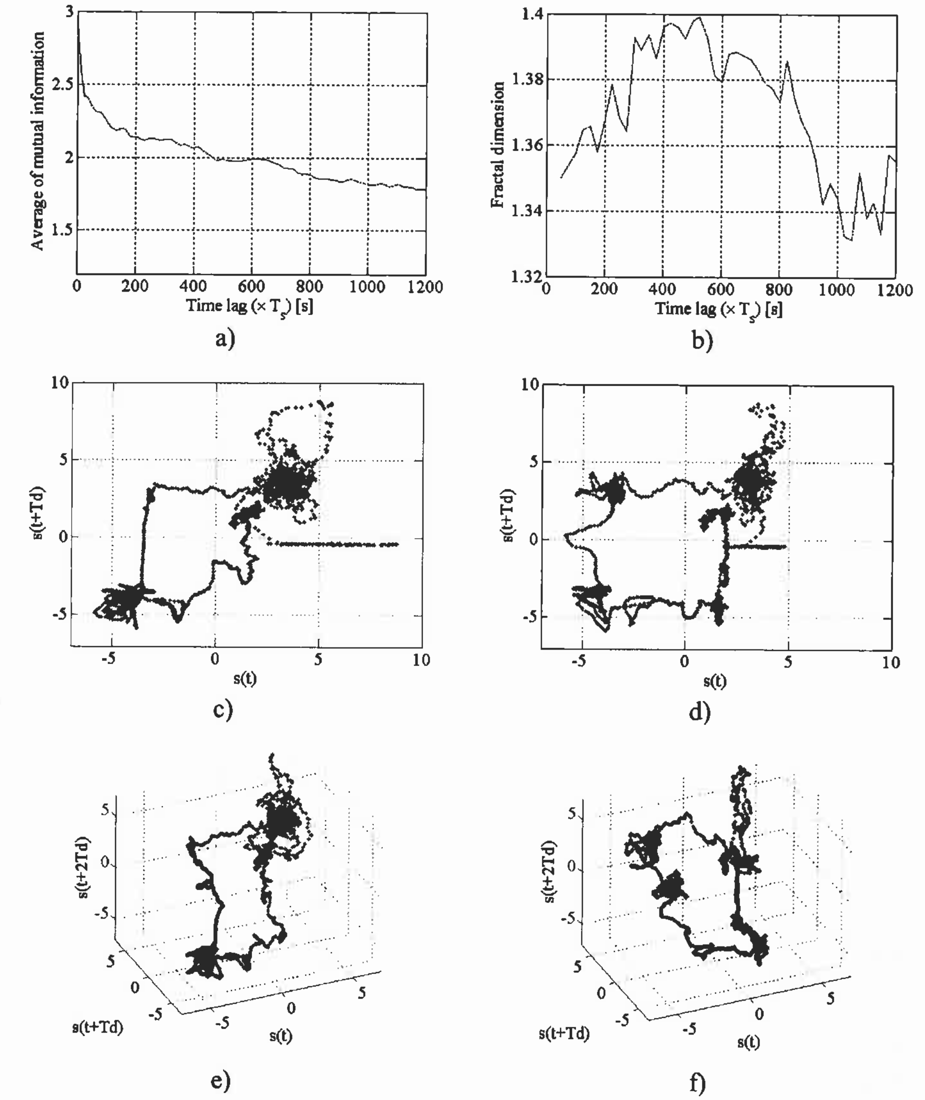


Figure 6. *Indexes of the electrical current of robot axis 3 for the experiment with the liquid container empty: a) smoothed version of I_{av} vs time lag; b) dim_{PPP} vs time lag; c) PPP for T_d = 150 samples (0.3s); d) PPP for T_d = 525 samples (1.05s); e) 3D PPS for T_d = 150 samples; f) 3D PPS for T_d = 525 samples*

Figure 7 shows the amplitude of the FFT of the electrical current of robot axis 3 for the case of having the container filled with a liquid. It is clear that model [1] yields a simple, but good, approximation. Therefore, a trendline is calculated and

superimposed over the signal in a frequency range larger than one decade ($3 < f < 90$ Hz). Its slope $m = -1.48$ corresponds to a fractional order behavior.

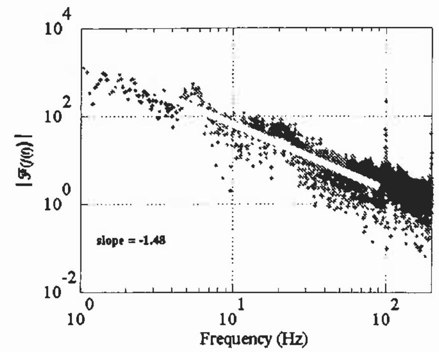


Figure 7. *The amplitude of the FFT for the electrical current of robot axis 3 (container filled with a liquid)*

Figure 8a depicts I_{av} (smoothed version) of the electrical current of robot axis 3 for the experiment with the container filled with a liquid when $C = 100$. The first local minimum occurs for time lag T_d = 300 (0.6 s), approximately. The corresponding index dim_{PPP} (Figure 8b) presents a global maximum at $(T_d, dim_{PPP}) = (400, 1.39)$, although at $(T_d, dim_{PPP}) = (300, 1.389)$ the value of dim_{PPP} is almost identical. Figures 8c–d) depict the PPP for these two different time lags. Practical evaluation reveals that the most adequate PPP is the one corresponding to Figure 8d because it is more unfolded. We can see this feature on $\{s(t), s(t+T_d)\} = \{4, -1\}$.

Figure 9a shows the amplitude of the FFT of the electrical current of robot axis 3 for the container filled with a solid. A trendline is calculated and superimposed over the signal in a frequency range larger than one decade ($3 < f < 90$ Hz). Its slope is $m = -1.48$. Figure 9b depicts I_{av} (smoothed version). The first local minimum occurs for time lag T_d = 300 (0.6 s), approximately. The corresponding dim_{PPP} (Figure 9c) presents a global maximum for T_d = 300. Figure 9d depicts the PPP.

The tests developed for others signals prove that the index dim_{PPP} is more sensitive comparing with I_{av}. Additionally, the chart of dim_{PPP} *versus* the time lag has a maximum corresponding to an adequate delay T_d. In resume, the index dim_{PPP} reveals to be more assertive than I_{av}, and to be an appropriate index for the time lag determination.

The PPP charts shown previously have a kind of "clouds" particularly at the corners. These clouds can hide superimposed curves. To unfold the curves we must find the proper embedding dimension. A key property of the embedding is that the mapping from the real space to the pseudo space is one-to-one. If trajectories cross each other in the PPP, then it is not an embedding (Feeny *et al.*, 2004). A deeper insight into the nature of this feature must be envisaged to understand the behavior of the PPP.

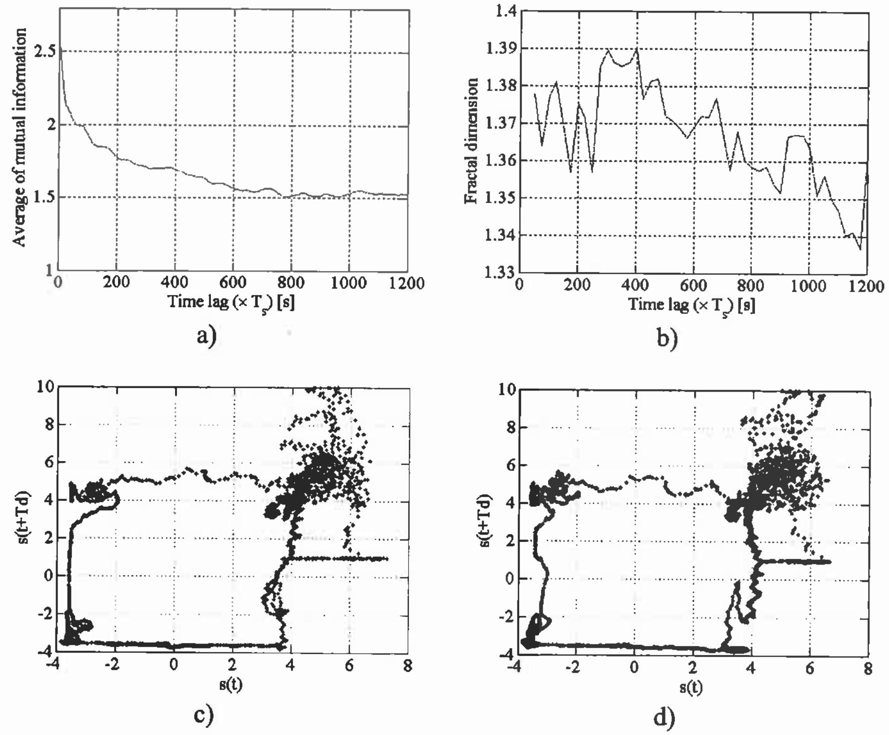


Figure 8. *Indexes of the electrical current of robot axis 3 for the experiment with the container filled with a liquid: a) smoothed version of I_{av} vs time lag; b) dim_{PPP} vs time lag; c) PPP for T_d = 300 samples (0.6 s); d) PPP for T_d = 400 samples (0.8 s)*

After analyzing, individually, the behavior of some electrical currents, we will explore some relationship between the variables. Figure 10 depicts the trendline slopes *versus* dim_{PPP} and T_d for the electrical current of all robot axis motors for the three cases of the container: empty, filled with a solid and filled with a liquid. Those fifteen points form the surfaces shown in figure 10 and the relationship between the three variables.

We verify the existence of a smooth curve linking the trendline slopes *versus* dim_{PPP} and T_d. Therefore, it seems that a relationship exists between the three variables. However, the establishment of an explicit analytical correlation is still to be investigated.

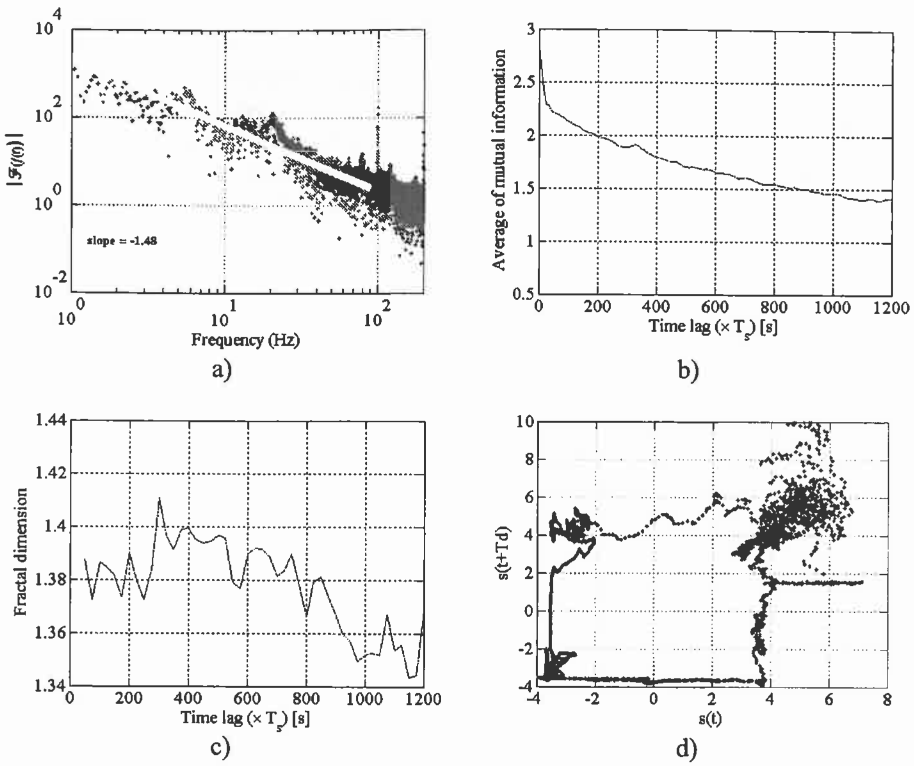


Figure 9. *Indices of the electrical current of robot axis 3 for the experiment with the container filled with a solid: a) FFT with a trendline; b) smoothed version of I_{av} vs time lag; c) dim_{PPP} vs time lag; d) PPP for T_d = 300 samples (0.6 s)*

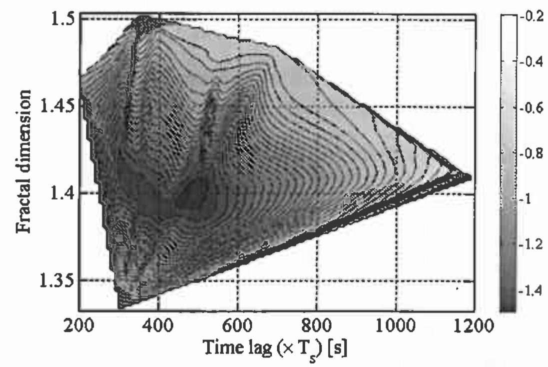


Figure 10. *Trendline slopes of the electrical motor currents spectra vs dim_{PPP} and time lag for the three cases of the container: empty, filled with a solid and filled with a liquid*

5. Conclusions

The spectrum of several robotic signals was approximated by trendlines. Based on the slope of the trendlines the fractional or integer order behavior was determined. For the PPP reconstruction for each signal we need the respective time lag. A new approach was proposed based on the fractal dimension. According to the tests the fractal dimension revealed to be an appropriate index leading to good results.

After analyzing individually the behavior of the electrical motor currents of a robotic system, was plotted the trendline slopes *versus* dim_{PPP} and the time lagged, for the three cases, namely container empty, filled with a solid and filled with a liquid. The plots show that all the points are located in a locus that demonstrate a relationship between the variables.

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