Instituto Superior Politécnico de Viseu
Escola Superior de Tecnologia de Viseu
Curso de Engenharia de Sistemas e Informática

Processamento Digital de Sinal

Aula 3

4.º Ano – 2.º Semestre

Manuel A. E. Baptista, Eng.º

Programa:

1. Introdução ao Processamento Digital de Sinal
2. Representação e Análise de Sinais
3. Estruturas e Projecto de Filtros FIR e IIR
4. Processamento de Imagem
5. Processadores Digitais de Sinal

SISTEMAS DE PROCESSAMENTO DIGITAL

Manuel A. E. Baptista
Ernesto R. Afonso

Bibliografia:

Processamento Digital de Sinal:

•Sanjit K. Mitra, "**Digital Signal Processing** – A computer based approach", McGraw Hill, 1998
Cota: 621.391 MIT DIG
•Roman Kuc, "**Introduction to Digital Signal Processing**", McGraw Hill, 1988.
Cota: 621.391 KUC INT
•Johnny R. Johnson, "**Introduction to Digital Signal Processing**", Prentice-Hall, 1989.
Cota: 621.391 JOH INT
G. Proakis, G. Manolakis, "**Digital Signal Processing – Principles, Algorithms Applications**", 3ª Ed, P-Hall, 1996.
Cota: 621.391 PRO DIG
•James V. Candy, "**Signal Processing – The modern Approach**", McGraw-Hill, 1988
Cota: 621.391 CAN SIG
•Mark J. T., Russel M., "**Introduction to DSP – A computer Laboratory Textbook**", John Wiley & Sons, 1992.
Cota: 621.391 SMI INT
•James H. McClellan e outros, "**Computer-Based Exercises - Signal Proc. Using Matlab 5**", Prentice-Hall, 1998.
Cota: 621.391 MCC COM

Processamento Digital de Imagem:

•Rafael C. Gonzalez & Richard E. Woods, "**Digital Image Processing** ", Prentice Hall, 2ª Ed., 2002.
Cota: 681.5 GON DIG.
•I. Pittas H. McClellan e outros, "Digital Image Processing Algorithms and Applications", John Wiley & Sons, 2000.
Cota: 621.391 PIT.
•William K. Pratt, "**Digital image processing**", John Wiley, 2ª Ed, 1991.
Cota: 681.5 PRA DIG
•Bernd Jähne, "**Digital image processing : concepts, algorithms, and scientific applications**", Springer, 1997.
Cota: 681.5 JAH

SISTEMAS DE PROCESSAMENTO DIGITAL

Manuel A. E. Baptista
Ernesto R. Afonso

Avaliação:

A avaliação é composta pela componente teórica e componente prática ponderadas da seguinte forma:

<u>Classificação Final</u> = 80% * Frequência ou exame + 20% * Prática

O acesso ao exame não está condicionado embora não tenha função de melhoria, ou seja, se o aluno entregar a prova de exame, será essa a classificação a utilizar no cálculo da média final independentemente da nota da prova de frequência obtida.

A avaliação prática é constituída por trabalhos laboratoriais a executar em MATLAB

Manuel A. E. Baptista
Ernesto R. Afonso

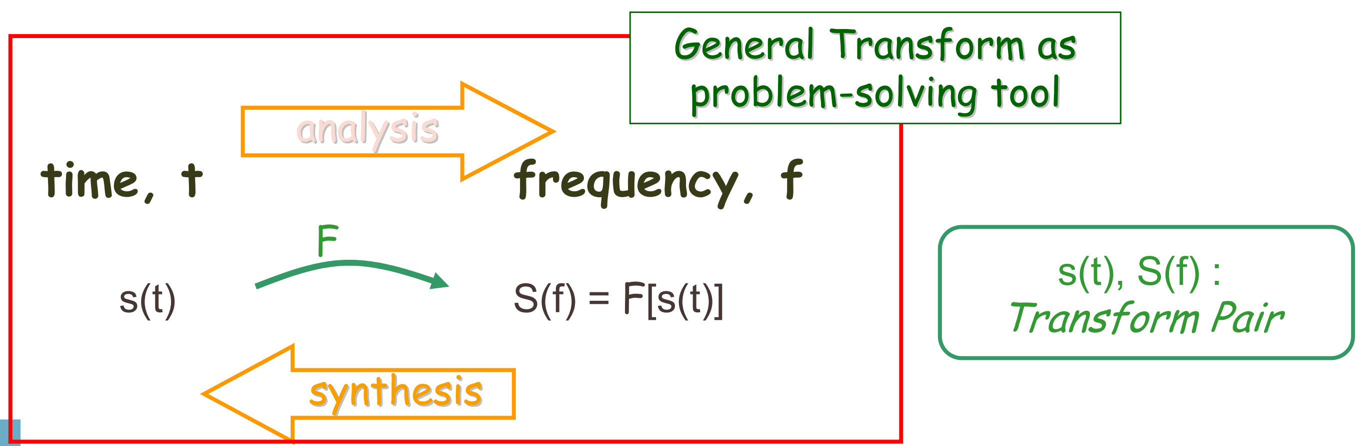
2. Representação e Análise de Sinais

- **Análise de Fourier - Sumário**
  - Análise na Frequência: Uma ferramenta poderosa…
  - Análise de Fourier - Ferramentas
  - Série Contínua de Fourier (FS)
  - Série Discreta de Fourier (DFS)
  - Exemplos

Manuel A. E. Baptista
Ernesto R. Afonso

Análise de Fourier: Porquê?

- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns. (PDE).
- Powerful & complementary to time domain analysis techniques.
- Several transforms in DSPing: Fourier, Laplace, z, etc.



Manuel A. E. Baptista
Ernesto R. Afonso

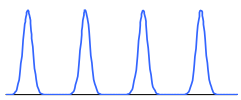
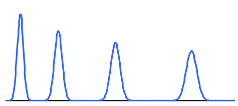
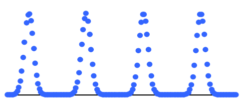
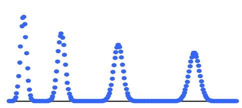
Análise de Fourier - Aplicações

Applications wide ranging and ever present in modern life

- *Telecomms* - GSM/cellular phones,
- *Electronics/IT* - most DSP-based applications,
- *Entertainment* - music, audio, multimedia,
- *Accelerator control* (tune measurement for beam steering/control),
- *Imaging, image processing,*
- *Industry/research* - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design,
- *Medical* - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis,
- *Speech analysis* (voice activated "devices", biometry, …).

Manuel A. E. Baptista
Ernesto R. Afonso

Análise de Fourier - ferramentas

Sinal de Entrada no Tempo				Espectro de Frequência	
	Contínuo	Periódico *(período T)*	FS	Discreto	$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-jk\omega t} dt$
		Aperiódico	FT	Continuo	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt$
	Discreto	Periódico *(período T)*	DFS**	Discreto	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$
		Aperiódico	DTFT	Continuo	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$
			DFT**	Discreto	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$

Nota: ***$j = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = No. of samples***

** ***Calculado via FFT***

Um pouco de história …

- Astronomic predictions by Babylonians/Egyptians likely via trigonometric sums.
- **1669**: Newton stumbles upon light spectra (*specter* = ghost) but fails to recognise "frequency" concept (*corpuscular* theory of light, & no waves).
- **18th century**: two outstanding problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation). **BUT** peers' consensus is that sum of sinusoids *only* represents smooth curves. Big blow to utility of such sums for all but Fourier ...
- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines. Strong criticism by peers blocks publication. Work published, 1822 (*"Theorie Analytique de la chaleur"*).

Manuel A. E. Baptista
Ernesto R. Afonso
Departamento de Informática

Um pouco de história …(2)

- **19th / 20th century**: two paths for Fourier analysis - Continuo & Discreto.

CONTINUO

→ Fourier extends the analysis to arbitrary function (Fourier Transform).

→ Dirichlet, Poisson, Riemann, Lebesgue address FS convergence.

→ Other FT variants born from varied needs (ex.: Short Time FT- speech analysis).

DISCRETO: Fast calculation methods (FFT)

→ **1805**- Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).

→ **1965**- IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).

→ Other DFT variants for different applications (ex.: Warped DFT- filter design & signal compression).

→ FFT algorithm refined & modified for most computer platforms.

Manuel A. E. Baptista
Ernesto R. Afonso
Departamento de Informática

Série de Fourier (FS)

A periodic function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

síntese

$$s(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k \cdot \cos(k\,\omega\,t) - b_k \cdot \sin(k\,\omega\,t) \right]$$

For all t but discontinuities

a_0, a_k, b_k : Fourier coefficients.

k: harmonic number,

T: period, $\omega = 2\pi/T$

analise

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t)dt$$

(signal average over a period, i.e. DC term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\;\omega\,t)\,dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\;\omega\,t)\,dt$$

Note: $\{\cos(k\omega t), \sin(k\omega t)\}_k$ form orthogonal base of function space.

* see next slide

Manuel A. E. Baptista
Ernesto R. Afonso
Departamento de Informática

Série de Fourier (FS) - Convergência

Dirichlet conditions

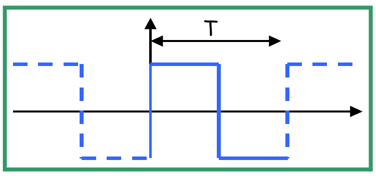
Em qualquer período:

(a) s(t) piecewise-continuous;

(b) s(t) piecewise-monotonic;

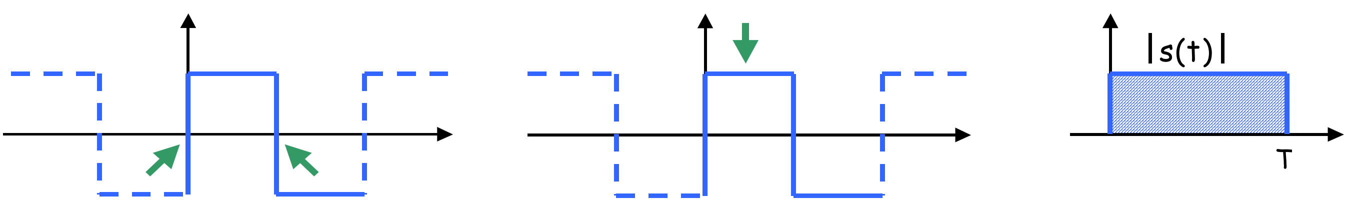
(c) s(t) absolutely integrable , $\int_0^T |s(t)|\,dt < \infty$

Exemplo: onda quadrada



Taxa de Convergência

if s(t) discontinuous then $|a_k| < M/k$ for large k (M>0)



(a) ☑ (b) ☑ (c) ☑

Manuel A. E. Baptista
Ernesto R. Afonso
Departamento de Informática

Série de Fourier (FS) - Análise

FS of odd* function: square wave.

$T = 2\pi \quad \Rightarrow \omega = 1$

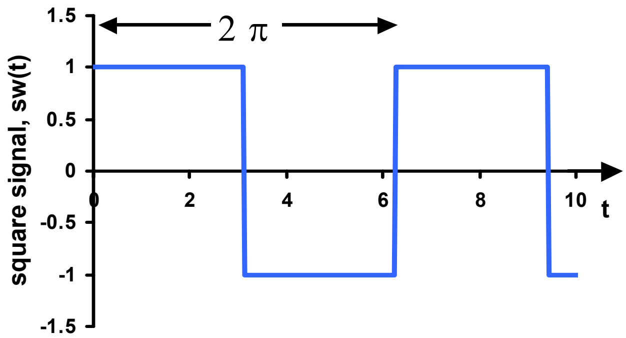
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\cos kt\,dt - \int_{\pi}^{2\pi}\cos kt\,dt\right\} = 0$$ *(função par)*

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\sin kt\,dt - \int_{\pi}^{2\pi}\sin kt\,dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot\{1-\cos k\pi\} =$$

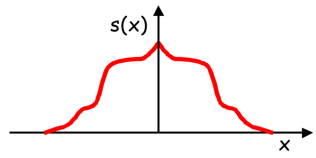
$$= \begin{cases} \dfrac{4}{k\cdot\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots$$

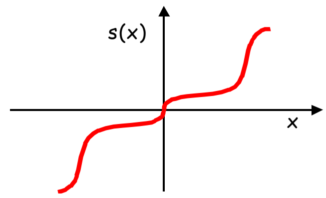


*** Funções Par & ímpar**

Par : s(-x) = s(x)



ímpar : s(-x) = -s(x)

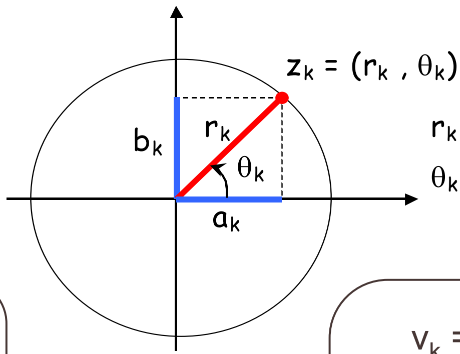


Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier (FS) – Análise (2)

Representações do Espectro de Fourier

$$s(t) = \sum_{k=0}^{\infty} v_k(t)$$

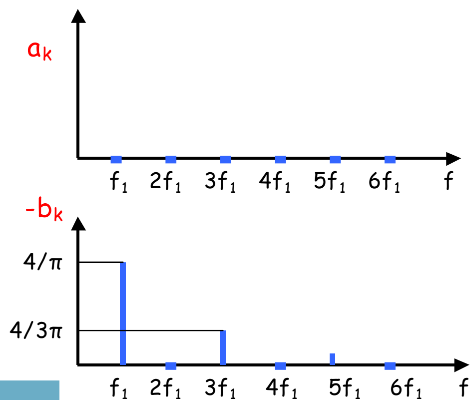


$r_k = \sqrt{a_k^2 + b_k^2}$

$\theta_k = \arctan(b_k/a_k)$

Rectangular

$v_k = \mathbf{a_k}\cos(\omega k\ t) - \mathbf{b_k}\sin(\omega k\ t)$



$\mathbf{r_K}$ = amplitude,

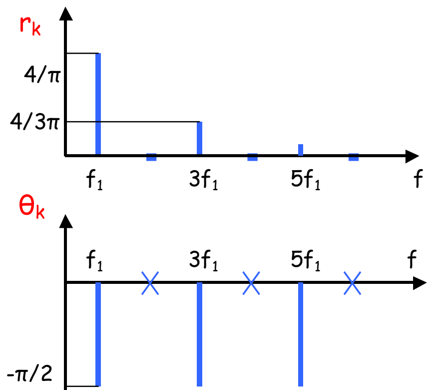
θ_K = fase

$f_K = k\ \omega/2\pi$

Fourier spectrum of square-wave.

Polar

$v_k = \mathbf{r_k}\cos(\omega k\ t + \theta_k)$

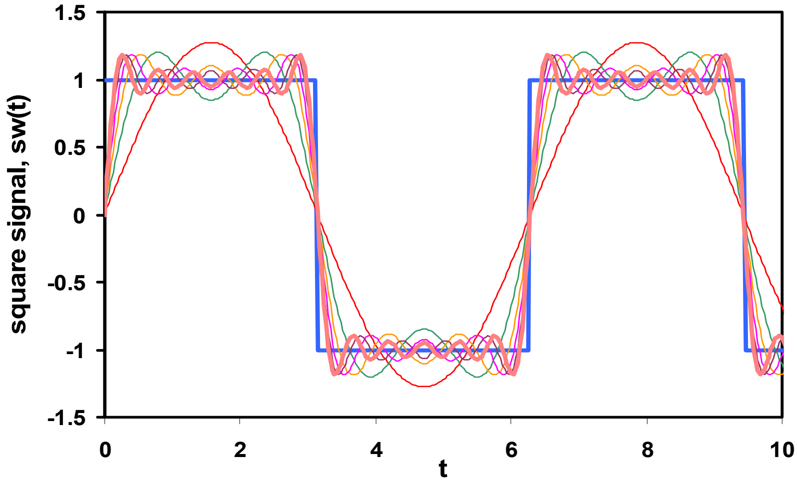


Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier (FS) – Síntese

Square wave reconstruction
from spectral terms

$$sw_9(t)=\sum_{k=1}^{9}\left[-b_k \cdot \sin(kt)\right]$$



Convergence may be slow (~1/k) - ideally need infinite terms.
Practically, series truncated when remainder below computer tolerance (⇒ *error*).
BUT … Gibbs' Phenomenon.

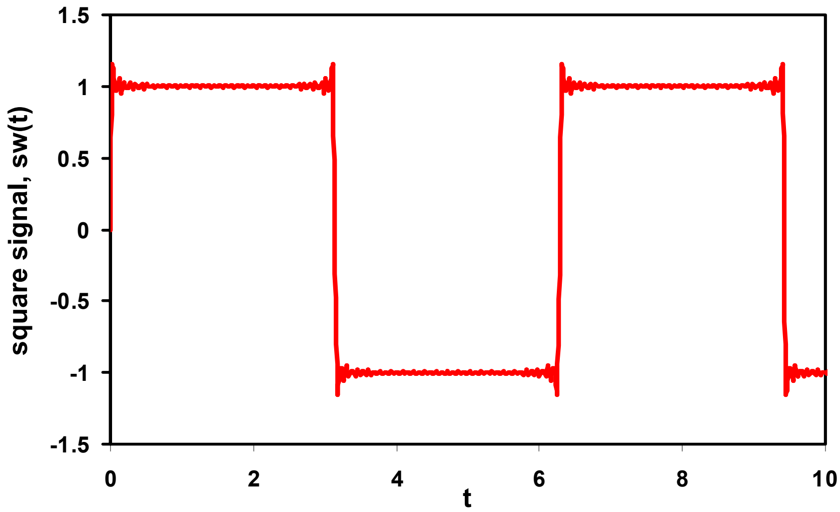
Manuel A. E. Baptista
Ernesto R. Afonso
Departamento de Informática

SISTEMAS DE PROCESSAMENTO DIGITAL

Série de Fourier (FS) – Fenómeno - Gibbs

Overshoot exist @ each
discontinuity

$$sw_{79}(t)=\sum_{k=1}^{79}\left[-b_k \cdot \sin(kt)\right]$$



- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot p_k-to-p_k = 8.95% of discontinuity magnitude.
  Just a minor annoyance.
- FS converges to (-1+1)/2 = 0 @ discontinuities, *in this case*.

Manuel A. E. Baptista
Ernesto R. Afonso
Departamento de Informática

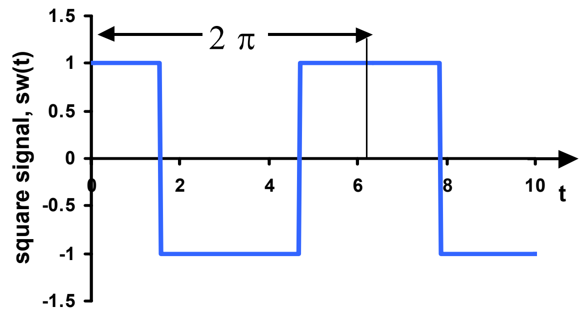
Série de Fourier (FS) – *Deslocamento no tempo*

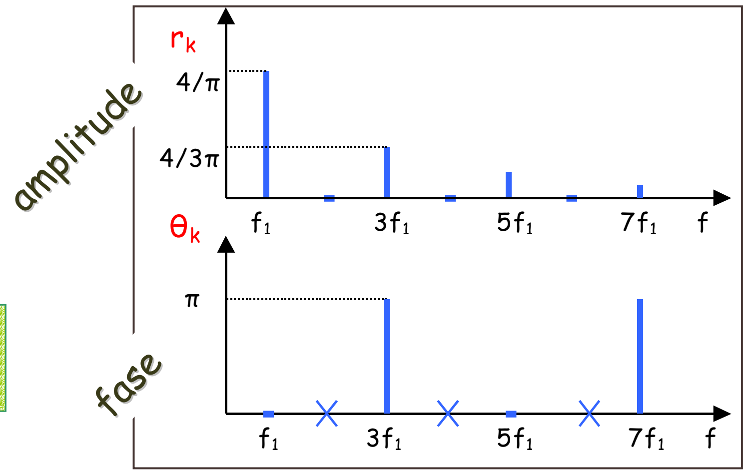
FS of even function:
π/2-advanced square-wave

$a_0 = 0$ *(média zero)*

$$a_k = \begin{cases} \frac{4}{k \cdot \pi}, & \text{k odd}, \quad k = 1, 5, 9... \\ -\frac{4}{k \cdot \pi}, & \text{k odd}, \quad k = 3, 7, 11... \\ 0, & \text{k even}. \end{cases}$$

$-b_k = 0$ *(função par)*





Nota: amplitudes unchanged ***BUT*** phases advance by k·π/2.

Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier (FS) - Complexa

Euler's notation:

$e^{-jt} = (e^{jt})^* = \cos(t) - j \cdot \sin(t)$ ⇒ "phasor"

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2} \qquad \sin(t) = \frac{e^{jt} - e^{-jt}}{2 \cdot j}$$

análise

$$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-jk\omega t} dt$$

síntese

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

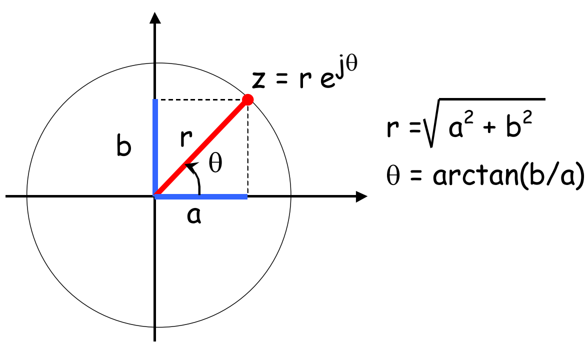
Complex form of FS (Laplace 1782). Harmonics c_k separated by Δf = 1/T on frequency plot.

Nota: $c_{-k} = (c_k)^*$

Link to FS real coeffs.

$$c_0 = a_0$$

$$c_k = \frac{1}{2} \cdot (a_k + j \cdot b_k) = \frac{1}{2} \cdot (a_{-k} - j \cdot b_{-k})$$



$r = \sqrt{a^2 + b^2}$

θ = arctan(b/a)

Manuel A. E. Baptista
Ernesto R. Afonso

Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier (FS) - Propriedades

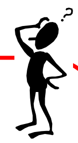
	Tempo	Frequência
Homogeneidade	$a \cdot s(t)$	$a \cdot S(k)$
Aditividade	$s(t) + u(t)$	$S(k)+U(k)$
Linearidade	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(k)+b \cdot U(k)$
Inversão em t	$s(-t)$	$S(-k)$
Multiplicação	$s(t) \cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolução	$\frac{1}{T} \cdot \int_0^T s(t-\bar{t}) \cdot u(\bar{t})\, d\bar{t}$	$S(k) \cdot U(k)$
Deslocamento em t	$s(t-\bar{t})$	$e^{-j\frac{2\pi k \cdot \bar{t}}{T}} \cdot S(k)$
Deslocamento em f	$e^{+j\frac{2\pi m t}{T}} \cdot s(t)$	$S(k - m)$

SISTEMAS DE PROCESSAMENTO DIGITAL

Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier (FS) - "*singularidades*"

Base ortonormal



Fourier components $\{u_k\}$ form orthonormal base of signal space:

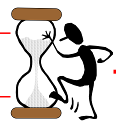
$u_k = (1/\sqrt{T}) \exp(jk\omega t)$ ($|k| = 0,1\ 2, \ldots +\infty$) **Def.:** Internal product $\otimes$:

$$u_k \otimes u_m = \int_0^T u_k \cdot u_m^* \, dt$$

$u_k \otimes u_m = \delta_{k,m}$ (1 if k = m, 0 otherwise). (Remember $(e^{jt})^* = e^{-jt}$)

Then $c_k = (1/\sqrt{T})\, s(t) \otimes u_k$ i.e. $(1/\sqrt{T})$ times *projection* of signal s(t) on component u_k

Frequências negativas & Inversão no Tempo



$k = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots + \infty$, $\omega_k = k\omega$, $\phi_k = \omega_k t$, phasor turns anti-clockwise.

Negative k $\Rightarrow$ phasor turns clockwise (negative phase ϕ_k), equivalent to negative time t,

$\Rightarrow$ time reversal.

Cuidado: phases important when combining several signals!

Departamento de Informática
www.estv.ipv.pt

Série de Fourier (FS) - energia

***Energia Média W*:** $W = \frac{1}{T}\int_{0}^{T} |s(t)|^2\, dt \equiv s(t) \otimes s(t)$

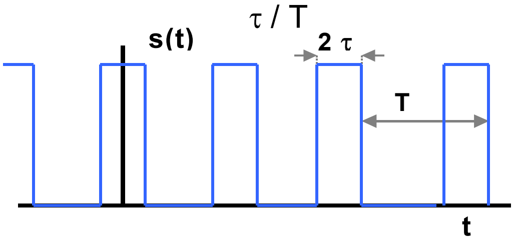
Teorema de Parseval

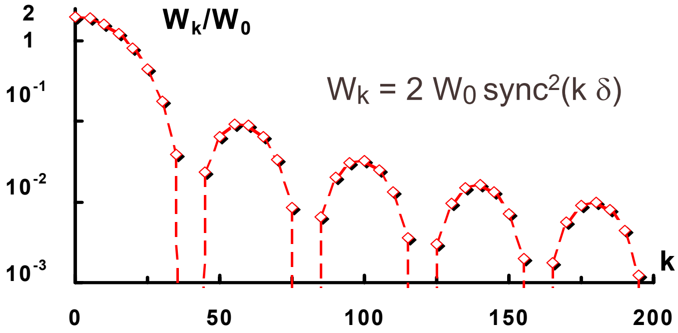
$$W = \sum_{k=-\infty}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

- **FS convergence ~1/k**
  - $\Rightarrow$ *lower frequency terms* $W_k = |c_k|^2$ *carry most power.*
- **W_k *vs.* ω_k: Power density spectrum.**

Exemplo

Trem de Impulsos, duty cycle $\delta = 2\ \tau / T$





$b_k = 0$ $\quad a_0 = \delta\, s_{MAX}$

$a_k = 2\delta s_{MAX}\, \mathrm{sync}(k\,\delta)$

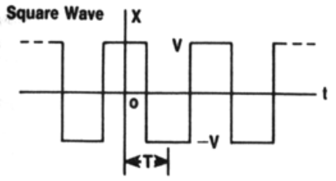
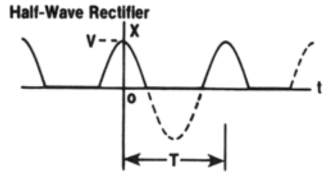
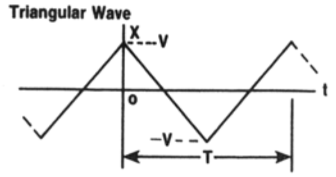
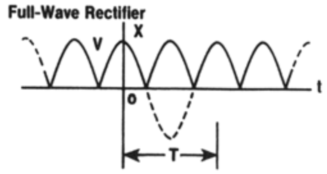
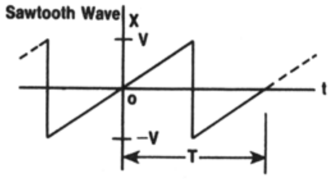
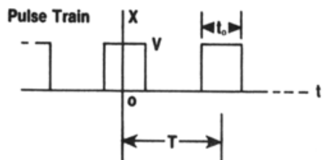
$W_0 = (\delta\, s_{MAX})^2$

$\mathrm{sync}(u) = \sin(\pi u)/(\pi u)$

$$W = W_0 \cdot \left\{1 + \sum_{k=1}^{\infty} \frac{W_k}{W_0}\right\}$$

Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier (FS) - Formas de ondas mais importantes

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$	Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Square Wave 	$x(t) = \frac{4V}{\pi}\left(\cos\omega_o t - \frac{1}{3}\cos 3\omega_o t + \frac{1}{5}\cos 5\omega_o t - \frac{1}{7}\cos 7\omega_o t + \ldots\right)$	Half-Wave Rectifier 	$x(t) = \frac{V}{\pi}\left(1 + \frac{\pi}{2}\cos\omega_o t + \frac{2}{3}\cos 2\omega_o t - \frac{2}{15}\cos 4\omega_o t + \frac{2}{35}\cos 6\omega_o t - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t \ldots\right)$ n even
Triangular Wave 	$x(t) = \frac{8V}{\pi^2}\left(\cos\omega_o t + \frac{1}{9}\cos 3\omega_o t + \frac{1}{25}\cos 5\omega_o t + \ldots\right)$	Full-Wave Rectifier 	$x(t) = \frac{2V}{\pi}\left(1 + \frac{2}{3}\cos 2\omega_o t - \frac{2}{15}\cos 4\omega_o t + \frac{2}{35}\cos 6\omega_o t - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t \ldots\right)$ n even
Sawtooth Wave 	$x(t) = \frac{2V}{\pi}\left(\sin\omega_o t - \frac{1}{2}\sin 2\omega_o t + \frac{1}{3}\sin 3\omega_o t - \frac{1}{4}\sin 4\omega_o t + \ldots\right)$	Pulse Train  1754-16 $k = t_o/T$	$x(t) = V\left[k + \frac{2}{\pi}\left(\sin k\pi \cos\omega_o t + \frac{1}{2}\sin 2k\pi \cos 2\omega_o t + \ldots + \frac{1}{n}\sin nk\pi \cos n\omega_o t + \ldots\right)\right]$ $k = t_o/T$

Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier Discreta (DFS)

Band-limited signal s[n], period = N.

DFS generate periodic c_k with same signal period

DFS definida como:

análise

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

Nota: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ mesmo período N

i.e. time periodicity propagates to frequencies!

Ortogonalidade nas DFS:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

Kronecker's delta

síntese

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

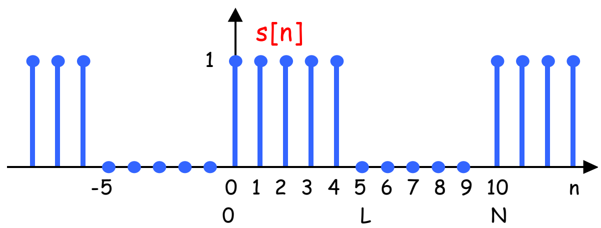
Síntese: finite sum $\Leftarrow$ band-limited s[n]

N consecutive samples of s[n] completely describes in time or frequency domains.

Manuel A. E. Baptista
Ernesto R. Afonso

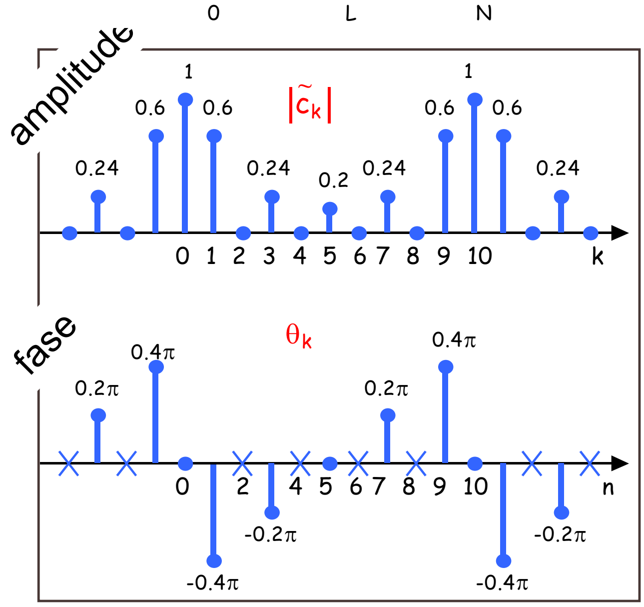
Série de Fourier Discreta (DFS) - Análise

DFS duma onda quadrada discreta de amplitude 1V



s[n]: período N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \dfrac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\[2ex] \dfrac{e^{-j\frac{\pi k (L-1)}{N}}}{N} \cdot \dfrac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, & \text{otherwise} \end{cases}$$



Discrete signals $\Rightarrow$ periodic frequency spectra.

Compare to continuous rectangular function!!!

Manuel A. E. Baptista
Ernesto R. Afonso

Série de Fourier Discreta (DFS) - Propriedades

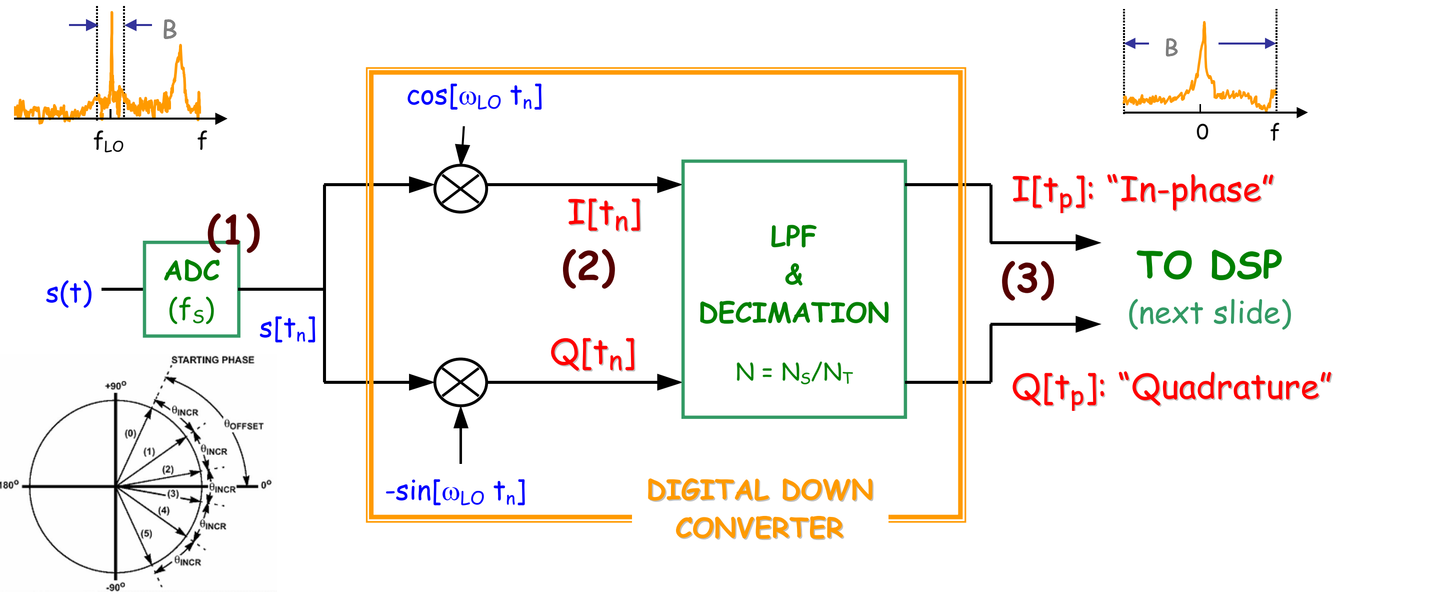
	Tempo	Frequência
Homogeneidade	$a\cdot s[n]$	$a\cdot S(k)$
Aditividade	$s[n] + u[n]$	$S(k)+U(k)$
Linearidade	$a\cdot s[n] + b\cdot u[n]$	$a\cdot S(k)+b\cdot U(k)$
Multiplicação	$s[n]\cdot u[n]$	$\frac{1}{N}\cdot\sum_{h=0}^{N-1} S(h)U(k-h)$
Convolução	$\sum_{m=0}^{N-1} s[m]\cdot u[n-m]$	$S(k)\cdot U(k)$
Deslocamento em t	$s[n-m]$	$e^{-j\frac{2\pi k\cdot m}{T}}\cdot S(k)$
Deslocamento em f	$e^{+j\frac{2\pi h t}{T}}\cdot s[n]$	$S(k-h)$

Manuel A. E. Baptista
Ernesto R. Afonso

SISTEMAS DE PROCESSAMENTO DIGITAL

Série de Fourier Discreta (DFS) – Análise: DDC + ...

s(t) periodic with period T_{REV} (ex: particle bunch in "racetrack" accelerator)



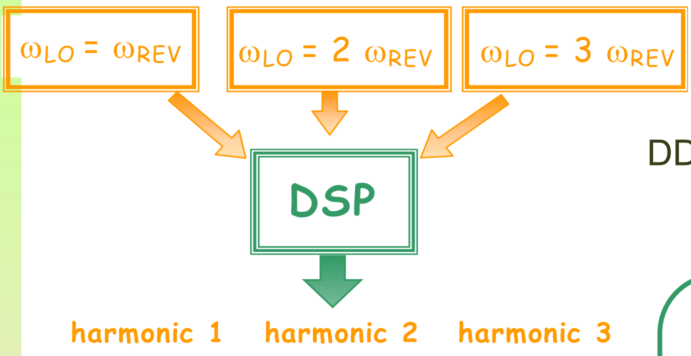
(1) $t_n = n/f_S$, n = 1, 2 .. N_S , N_S = No. samples

(2) $I[t_n]+j\,Q[t_n] = s[t_n]\,e^{-j\omega_{LO}t_n}$

(3) $I[t_p]+j\,Q[t_p]$ p = 1, 2 .. N_T , N_s / N_T = decimation. (Down-converted to baseband).

Manuel A. E. Baptista
Ernesto R. Afonso

... + DSP



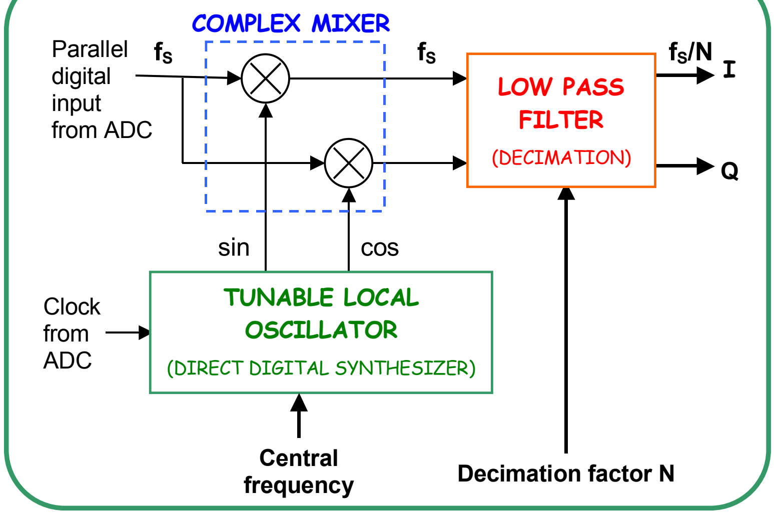
DDCs with different f_{LO} yield more DFS components

Coeficientes de Fourier a_{k^*}, b_{k^*}

$$a_{k^*} = \frac{1}{N_T} \cdot \sum_{p=1}^{N_T} I_p \qquad b_{k^*} = -\frac{1}{N_T} \cdot \sum_{p=1}^{N_T} Q_p$$

Harmónico $k^* = \omega_{LO}/\omega_{REV}$

Exemplo: Real-life DDC



Manuel A. E. Baptista
Ernesto R. Afonso